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# HEAT TRANSFER, TEMPERATURE RECOVERY AND SKIN FRICTION ON A FLAT PLATE WITH HYDROGEN RELEASE INTO A LAMINAR BOUNDARY LAYER

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**Abstract**—An analysis is presented for a two-component laminar boundary layer on a surface with zero pressure gradient and with mass release. The system of partial differential equations describing the velocity, temperature, and concentration field is converted by a similarity transformation into a system of integral equations. Numerical solutions are obtained on electronic digital computers for air flow with hydrogen injection, for Mach numbers 0 to 12, for free-stream temperatures 123 to 2000°R, and for wall-to-free stream temperature ratios from $\frac{1}{2}$ to 6. Solutions for the condition of zero temperature gradient in the boundary layer at and normal to the wall surface lead to temperature recovery factors. The results are utilized to investigate the validity and accuracy of engineering correlations presented in [15]. The proposed correlations describe the reduction in heat flux into the wall surface and in skin friction reasonably well. No simple correlations have been found for the dependence of hydrogen mass fraction at the surface on the mass release rate and for the temperature recovery factor. Values for a free-stream temperature of 123°R show generally the largest deviations from the rest of the data.

## NOMENCLATURE

| | |
|---|---|
| $a$, | sound velocity; |
| $c_f$, | local skin friction coefficient; |
| $c_p$, | specific heat at constant pressure; |
| $f$, | dimensionless stream function; |
| $h$, | enthalpy; |
| $k$, | thermal conductivity; |
| $\dot{m}$, | mass flow per unit area and time; |
| $q$, | heat flow per unit area and time; |
| $r$, | recovery factor; |
| $t$, | temperature; |
| $u$, | velocity component parallel to surface; |
| $v$, | velocity component normal to surface; |
| $x$, | co-ordinate along the body; |
| $y$, | co-ordinate normal to the body; |
| $D$, | diffusion coefficient; |
| $M_1$, | molecular weight of coolant gas; |
| $M_2$, | molecular weight of pure air; |
| $T$, | temperature ratio; |
| $W$, | mass fraction of foreign gas. |

### Greek symbols

| | |
|---|---|
| $\alpha$, | thermal diffusivity; |
| $\gamma$, | ratio of specific heats; |
| $\eta$, | dimensionless wall distance; |
| $\mu$, | dynamic viscosity; |
| $\nu$, | kinematic viscosity; |
| $\rho$, | density; |
| $\varphi_c$, | normalized mixture thermal capacity $(c_p/c_{p\infty})$; |
| $\varphi_{c12}$, | normalized pure-component thermal capacity difference $(c_{p1} - c_{p2})/c_{p\infty}$; |
| $\varphi_k$, | normalized mixture thermal conductivity $(k/k_\infty)$; |
| $\varphi_\mu$, | normalized mixture viscosity $(\mu/\mu_\infty)$; |
| $\varphi_\rho$, | normalized mixture density $(\rho/\rho_\infty)$; |
| $\psi$, | stream function. |

### Dimensionless parameters

| | |
|---|---|
| $c_f$, | local skin friction coefficient, $\left(\mu \frac{\partial u}{\partial y}\right)_w \Big/ (\frac{1}{2}\rho_\infty u_\infty^2)$; |
| $Le$, | Lewis number, $a/D$; |
| $M$, | Mach number, $u/a$; |
| $Pr_\infty$, | Prandtl number, $\mu_\infty c_{p\infty}/k_\infty$; |
| $r$, | temperature recovery factor, $(t_r - t_\infty)/(t_t - t_\infty)$; |
| $Re_{x,\infty}$, | Reynolds number, $\rho_\infty u_\infty x/\mu_\infty$; |
| $Re_x^{*\prime}$, | Reynolds number, $\rho^{*\prime} u_\infty x/\mu^{*\prime}$. |

### Subscripts

| | |
|---|---|
| 1, | refers to pure coolant; |
| 2, | refers to pure air; |

| | |
|---|---|
| $w$, | evaluated at wall conditions; |
| $i$, | $i$-th component; |
| $R$, | refers to radiation; |
| $r$, | refers to recovery conditions; |
| $\infty$, | outside the boundary layer; |
| 0, | at solid surface; |
| $s$, | refers to solid wall; |
| $t$, | refers to total state. |

Superscripts

| | |
|---|---|
| $*'$, | evaluated at the reference temperature, equation (11). |

## 1. INTRODUCTION

RECENT advances in high speed and space flight technology have focused attention on the aerodynamic heating problem. The search for effective cooling methods for the protection of re-entering vehicles against the intense heat generated by friction in the atmosphere has led to the development of "mass-transfer cooling". This name includes all methods in which a mass flow of coolant gas is created from the cooled surface into the hot boundary layer irrespective of the particular way in which this mass is released. In transpiration cooling, for example, the coolant gas is ejected through a porous wall into the boundary layer. In ablation cooling, the coolant gas is produced at the surface by sublimation or some chemical reaction of the wall material itself. In liquid film cooling, evaporation of a liquid from the surface of a film covering the cooled wall produces the coolant gas. The term mass-transfer cooling focuses attention on the conditions in the boundary layer itself and on the way they are influenced by the mass release at the surface.

An analysis of mass-transfer cooling thus has to deal with boundary layer flow of a more component—at least two component—gas mixture composed of the fluid in the main stream and of the coolant and has to investigate the momentum, energy, and mass transport within this layer. Such analyses were at first carried out with the assumption that the fluid properties can be considered as constant within the boundary layer [1, 2]. The results indicated that heat transfer to the wall surface can be considerably reduced by the "blocking effect" which the coolant gas produced by moving away from the wall surface. Subsequent calculations, which considered the variations of the physical properties with composition and temperature, indicated that gases with low molecular weight are especially effective as coolant, insofar as they produce the largest decrease of the heat flow into the wall surface at a fixed coolant mass release [3, 4, 5, 7]. Hydrogen, as the gas with the smallest molecular weight, appears, therefore, especially promising.

There are two types of flow which are well suited for an analysis: plane or rotationally symmetric stagnation flow and flat plate flow. Results obtained for these are valuable since they can be considered as limiting cases of the behavior of boundary layers along the surface of objects exposed to a subsonic or supersonic flow. Analyses for laminar boundary-layer flow over a surface with constant pressure (flat plate) and with hydrogen as coolant are reported in [5, 7]. Laminar stagnation-point flow has been treated in [8]. Numerical calculations in these reports have been made for a temperature outside the boundary layer which is close to 400°R. In the present paper, the calculations are extended to a wide range of boundary conditions including free-stream temperatures up to 2000°R, keeping in mind that the results of a "flat plate" analysis should also describe, with good approximation, conditions for boundary layers on slender objects or on blunt objects in regions where the pressure gradients are small. In such cases, the air outside the boundary layer has passed at some upstream location through a shock and has been heated to a high temperature, especially in hypersonic flow. The results of the analysis are used to investigate the validity and accuracy of engineering correlations which have been proposed in [15] and which are in widespread use today.

It is assumed in this paper that no dissociation or chemical reactions occur within the boundary layer. In reality, such reactions will always be present at the higher temperature levels. Their inclusion in the analysis, however, is hampered by insufficient knowledge of chemical reaction kinetics and of the transport properties involved and has been included in boundary layer analyses only for gas mixtures with a Lewis number equal or close to one, for example in [9]. This condition is, unfortunately, not too well fulfilled for hydrogen–air mixtures. Therefore, only approximate estimates on the effect of reactions on the heat transfer in such a boundary layer can be made at present and more exact analysis has to await the accumulation of more knowledge on the above-mentioned parameters.

Only a few examples of the results of this analysis can be presented in this paper. Complete information will be contained in a later publication.

## 2. ANALYTICAL FORMULATION

### *Basic equations*

The equations for the steady, laminar, binary mixture boundary layer on a surface with constant pressure in flow direction (flat plate) may be written as follows:

Continuity:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0; \qquad \text{(I)}$$

Momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0; \qquad \text{(II)}$$

Diffusion:

$$\rho u \frac{\partial W}{\partial x} + \rho v \frac{\partial W}{\partial y} - \frac{\partial}{\partial y}\left(\rho\, D_{12} \frac{\partial W}{\partial y}\right) = 0; \qquad \text{(III)}$$

Energy:

$$\rho\, c_p\, u \frac{\partial t}{\partial x} + \rho\, c_p\, v \frac{\partial t}{\partial y} - \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) - \mu \left(\frac{\partial u}{\partial y}\right)^2 + \rho\, D_{12}\,(c_{p2} - c_{p1}) \frac{\partial t}{\partial y} \frac{\partial W}{\partial y} = 0. \qquad \text{(IV)}$$

These equations are based, apart from the standard boundary layer arguments, on the assumption that the mixture is inert and that thermal diffusion is negligible. The physical properties which enter into the equation system (I) to (IV) are assumed to be given functions of mass concentration $W$ and temperature $t$. The specific formulas are discussed in detail in the section on Physical Properties. All symbols are defined in the List of Nomenclature.

The energy equation has been written in various recent publications in enthalpy as dependent parameter rather than in temperature. This simplifies the equation considerably for a gas mixture with a Lewis number equal to one. It offers, however, no advantage for a gas mixture with a different Lewis number. Therefore, equation (IV) is based on temperature.

$W$ denotes the mass fraction of coolant gas (hydrogen) released at the surface. In the following numerical calculations, the mainstream will be assumed to be air and the mass fraction of coolant outside the boundary to be zero.

Two sets of boundary conditions are considered. They differ only in the boundary conditions imposed on the energy equation, equation (IV), for which either the wall temperature or a temperature gradient zero in the fluid at the wall is prescribed. Specifically, we state the following:

$$\text{for } y = 0: \quad u = 0$$
$$v = v_w(x)$$
$$t = t_w \text{ or } \left(\frac{\partial t}{\partial y}\right)_{y=0} = 0$$
$$\text{for } y \to \infty: \quad \lim u(x,y) = u_\infty = \text{const.}$$
$$\lim W(x,y) = 0$$
$$\lim t(x,y) = t_\infty = \text{const.}$$

Whenever $t = t_w$ at $y = 0$ is imposed, we shall speak of the cooled wall case. If $(\partial t/\partial y)_{y=0} = 0$ is imposed, we shall term this the recovery situation.

On the assumption that at the surface the net mass flow of air is zero, we obtain

$$(1 - W_w)\, \rho_w\, v_w = -\, \rho_w\, D_{12w} \left(\frac{\partial W}{\partial y}\right)_w. \qquad (1)$$

Thus, either $W_w$ or $v_w$ may be prescribed independently, the other being determined by the above side condition.

With the system of equations (I) to (IV), the boundary conditions, and equation (1) the problem is completely defined. This implies that the way that the coolant mass flow leaving the wall surface is created (by transpiration, ablation, evaporation) has no influence on the boundary layer development as long as the wall temperature, $t_w$, and the injection parameter, $v_w$, are prescribed. For a better understanding of the parameters in which the results of the analysis are presented, however, it is useful to consider, at this time, a general energy balance on the interface between the wall and the fluid, as shown in Fig. 1. The wall may be a porous structure through which the coolant gas is blown, or it may be a solid at the surface of which the coolant gas is generated by sublimation.

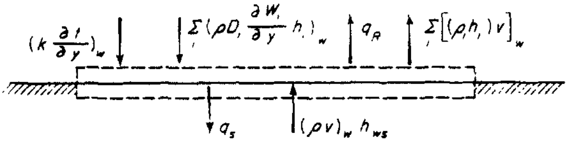

FIG. 1.

The net heat input, $q_s$, to the solid wall material, is given for an $i$-component gas mixture by the relation:

$$q_s = \left(k \frac{\partial t}{\partial y}\right)_w + \left[\sum_i \rho D_i \frac{\partial W_i}{\partial y} h_i\right]_w + (\rho v)_w h_{ws} - [\sum_i (\rho_i h_i) v]_w - q_R.$$

The above expression states that energy is transported from the gas mixture to the interface by conduction $[k (\partial t/\partial y)]$ and by diffusion $[\sum_i \rho D_i (\partial W_i/\partial y) h_i]$. Simultaneously, there occurs an energy flux from the inside of the wall towards the interface at the rate $(\rho v)_w h_{ws}$. In transpiration cooling this energy flux is transported by the coolant gas as it moves toward the wall surface. In ablation cooling $(\rho v)_w$ is the rate with which the interphase recedes into the solid wall. The enthalpy $h_{ws}$ is equal to the enthalpy of the coolant gas, as it passes the surface, for transpiration cooling, and equal to the enthalpy of the solid or liquid in ablation or evaporation cooling. Finally, energy is transported away from the interface by convection $[\sum_i (\rho_i h_i) v]$ and radiation $(q_R)$. $q_s$ is the heat conducted away from the surface in the solid wall material. For the binary system at hand, the above equation reads

$$q_s = \left(k \frac{\partial t}{\partial y}\right)_w + \left[\rho D_{12} \frac{\partial W}{\partial y} (h_1 - h_2)\right]_w + (\rho v)_w h_{ws} - [(\rho_1 h_1 + \rho_2 h_2) v]_w - q_R.$$

The total mass fluxes of hydrogen (1) and air (2) are given by

$$\dot{m}_{1w} = -\left(\rho D_{12} \frac{\partial W}{\partial y}\right)_w + \rho_{1w} v_w$$

and

$$\dot{m}_{2w} = \left(\rho D_{12} \frac{\partial W}{\partial y}\right)_w + \rho_{2w} v_w = 0.$$

Hence

$$\dot{m}_w = \dot{m}_{1w} + \dot{m}_{2w} = (\rho v)_w = \rho_{1w} v_w - \left(\rho D_{12} \frac{\partial W}{\partial y}\right)_w.$$

Substitution of these expressions into the above equation yields

$$q_s = \left(k \frac{\partial t}{\partial y}\right)_w - (\rho v)_w (h_{1w} - h_{ws}) - q_R. \quad (2)$$

The second term on the right-hand side vanishes for transpiration cooling. The first term on the right-hand side of equation (2) is determined by the boundary layer equations and their boundary conditions. It will be denoted by

$$q = -\left(k \frac{\partial t}{\partial y}\right)_w \quad (3)$$

and is presented in the later part of this paper. The rest of the terms in equation (2) depend on the cooling method, on the radiative exchange process, and on the temperature history of the wall and has to be calculated from considerations outside the scope of this paper.

In the recovery problem, by definition,

$$q = -\left(k \frac{\partial t}{\partial y}\right)_w = 0.$$

A recovery factor $r$ is also defined by

$$r = \frac{t_r - t_\infty}{t_t - t_\infty} = \frac{T_r - 1}{T_t - 1}. \quad (4)$$

The recovery factor is a parameter which was found useful in the analysis of heat transfer from a high velocity gas to a solid surface, and in transpiration cooling in a constant property boundary layer [1].

*Similarity transformation*

We seek a class of "similar" solutions. Introduction of the stream function $\psi$ defined by

$$u = \frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial y}$$

and

$$v = -\frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial x}$$

satisfies the continuity equation identically. The similarity variable

$$\eta = \tfrac{1}{2}\sqrt{\left(\frac{u_\infty}{\nu_\infty x}\right)}\int_0^y \frac{1}{\varphi_\mu}\,\mathrm{d}y \qquad (5)$$

and the assumption

$$\psi = \sqrt{(u_\infty/\nu_\infty x)}\,f(\eta)$$

transforms the equations (II) to (IV) into the following system:

Momentum:

$$\left[\left(\frac{f'}{\varphi_\mu\varphi_\rho}\right)'\right]' + f\left(\frac{f'}{\varphi_\mu\varphi_\rho}\right)' = 0; \qquad \text{(II')}$$

Diffusion:

$$\left(\frac{W'}{Sc}\right)' + fW' = 0; \qquad \text{(III')}$$

Energy:

$$\frac{1}{Pr_\infty}\left(\frac{\varphi_k T'}{\varphi_\mu}\right)' + \varphi_{c_{12}}\, Sc\, W'\, T' + \varphi_c f T'$$
$$+ \tfrac{1}{4}(\gamma_\infty - 1)\, M_\infty^2\left[\left(\frac{f'}{\varphi_\mu\varphi_\rho}\right)'\right]^2 = 0. \qquad \text{(IV')}$$

The properties have been made dimensionless by referring them to the respective property outside the boundary layer. The temperature ratio $T = t/t_\infty$ has been introduced. The primes denote differentiation with respect to $\eta$. The equation system (II'), (III'), (IV'), may also be thought of as being obtained from (I) to (IV) by a double transformation. One first introduces the Blasius variable

$$\eta_B = \tfrac{1}{2}y\sqrt{\left(\frac{u_\infty}{\nu_\infty x}\right)}$$

and transforms the resulting set of equations in $\eta_B$ a second time by introducing

$$\eta = \int_0^{\eta_B}\frac{1}{\varphi_\mu}\,\mathrm{d}\eta_B.$$

The result, with the proper reinterpretation of $f(\eta)$, is again the equation system (II') to (IV').

The boundary conditions become:

for $\eta = 0$: $f' = 0$,
$f = f_w = \text{const.}$,
$W = W_w$,
$T = T_w$ or $T' = 0$,

for $\eta \to \infty$: $\lim f' = 2$,
$\lim W = 0$,
$\lim T = 1$.

The side condition, equation (1), transforms to

$$f_w = \frac{W'_w}{Sc_w(1 - W_w)}. \qquad (7)$$

Again it should be noted that either $f_w$ or $W_w$ may be prescribed independently, but not both.

Equations (II') to (IV') comprise a set of nonlinear ordinary differential equations which must be solved simultaneously satisfying one of the sets of boundary conditions given above. The interdependence between (II'), (III') and (IV'), is strongly affected by the particular choice of property relations. For example, if the properties were assumed constant, (II') could be solved independently of (III') and (IV'). The solution of (II') would then be used to solve (III') and (IV'). Such procedure is evidently especially simple. For this reason, one seeks forms of the similarity variable $\eta$ which give, in the final working equations (II'') to (IV''), combinations of properties as little dependent of local mass concentration and temperature as possible. This appears to be accomplished with the choice of $\eta$ used in this paper at least for light-weight gases such as hydrogen or helium. In fact, the energy equations in the integral form (equations IV″ and IV‴) are in this way effectively uncoupled from the remaining equations since the property combinations are only mild functions of temperature. The sets of equations thus tend to behave as if the momentum and diffusion equations formed a simultaneous system with an independent energy equation.

### 3. METHOD OF SOLUTION

The equation system (II′) to (IV′) may be written as a set of integral equations. For the cooled wall one obtains:

Momentum:

$$f'(\eta) = C_1 \varphi_\mu \varphi_\rho \textstyle\int_0^\eta \exp\left(-\int_0^{\eta'} f \, d\eta''\right) d\eta', \quad \text{(II″)}$$

$$f(\eta) = f_w + \textstyle\int_0^\eta f' \, d\eta';$$

Diffusion:

$$W(\eta) = C_2 \textstyle\int_0^\eta Sc \exp\left(-\int_0^{\eta'} Sc\, f \, d\eta''\right) d\eta'; \quad \text{(III″)}$$

Energy:

$$T(\eta) = T_w + C_3 \int_0^\eta \frac{\varphi_\mu}{\varphi_k} \exp\left(-\textstyle\int_0^{\eta'} \lambda \, d\eta''\right) d\eta'$$

$$- \int_0^\eta \frac{\varphi_\mu}{\varphi_k} \exp\left(-\textstyle\int_0^{\eta'} \lambda \, d\eta''\right)$$

$$\left[\textstyle\int_0^{\eta'} g \exp\left(\int_0^{\eta''} \lambda \, d\eta'''\right) d\eta''\right] d\eta'. \qquad \text{(IV″)}$$

In these equations, $\lambda(\eta)$ and $g(\eta)$ are defined as

$$\lambda(\eta) = Pr_\infty \frac{\varphi_\mu}{\varphi_k} \left(\varphi_c f + \frac{\varphi_{c12}}{Sc} W'\right)$$

and

$$g(\eta) = Pr_\infty \left\{ \tfrac{1}{4} (\gamma_\infty - 1) M_\infty^2 \left[\left(\frac{f'}{\varphi_\mu \varphi_\rho}\right)'\right]^2 \right\}.$$

The constants $C_1$, $C_2$, $C_3$ are obtained from the conditions at infinity:

$$C_1 = \frac{2}{\int_0^\infty \exp\left(-\int_0^{\eta'} f \, d\eta''\right) d\eta'},$$

$$C_2 = \frac{-W_w}{\int_0^\infty Sc \exp\left(-\int_0^{\eta'} Sc\, f \, d\eta''\right) d\eta'},$$

$$C_3 = \frac{\left\{1 - T_w + \int_0^\infty \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right) \left[\int_0^{\eta'} g \exp\left(\int_0^{\eta''} \lambda \, d\eta'''\right) d\eta''\right] d\eta'\right\}}{\int_0^\infty \frac{\varphi_\mu}{\varphi_k} \exp\left(-\int_0^{\eta'} \lambda \, d\eta''\right) d\eta'}.$$

If the conductive heat flux at the surface is zero, equations (II″), (III″) remain unchanged. The energy equation (IV″) is replaced by

$$T = T_w - \int_0^\eta \frac{\varphi_\mu}{\varphi_k} \exp\left(-\textstyle\int_0^{\eta'} \lambda \, d\eta''\right)$$

$$\left[\textstyle\int_0^{\eta'} g \exp\left(\int_0^{\eta''} \lambda \, d\eta'''\right) d\eta''\right] d\eta' \qquad \text{(IV‴)}$$

where

$$T_w = 1 + \int_0^\infty \frac{\varphi_\mu}{\varphi_k} \exp\left(-\textstyle\int_0^{\eta'} \lambda \, d\eta''\right)$$

$$\left[\textstyle\int_0^{\eta'} g \exp\left(\int_0^{\eta'} \lambda \, d\eta'''\right) d\eta''\right] d\eta'.$$

The equation systems (II″) to (IV″) or (II″), (III″) and (IV‴) are now solved by successive approximations.

For cooled wall calculations, a straightforward iterative technique is satisfactory. The procedure is the following. Let the boundary layer profiles $W_0$, $T_0$, $f_0'$ and the blowing parameter $f_{w0}$ be the initial approximations. Substituting these into the right-hand side of equations (II″), (III″), (IV″) and utilizing equation (7) yields the next set of values $W_1$, $T_1$, $f_1'$, $f_{w1}$. This cycle is repeated until

$$\left.\begin{aligned} &\left|\textstyle\int_0^\infty (W_{i+1} - W_i) \, d\eta\right| < \delta \\ &\left|\textstyle\int_0^\infty (T_{i+1} - T_i) \, d\eta\right| < \delta \\ &\left|\textstyle\int_0^\infty (f_{i+1}' - f_i') \, d\eta\right| < \delta \\ &\left|\textstyle\int_0^\infty (f_{wi+1} - f_{wi}) \, d\eta\right| < \delta \end{aligned}\right\} \quad (8)$$

where $\delta$ was arbitrarily taken as $10^{-4}$. The subscript $i$ denotes here the $i$-th iteration. It may be shown that the uniform convergence of the above integrals constitutes a sufficient condition for the convergence of the integral equations.

In the recovery problem, the solution method is modified as follows. Let $W_0$, $T_0$, $f_0'$, $f_{w0}$, be the set of initial values. Then the equations (II″), (III″), (IV‴), together with equation (7), yield the set of values $W_1$, $T_1$, $f_1'$, $f_{w1}$. We consider $W_1$, $f_1'$, $f_{w1}$ to be the next approximations. The new approximation to $T$, however, is not $T_1$, but a certain perturbed set of values, $T_1^{\circledast}$, given by

$$T_1^{\circledast} = T_0 + \sigma(T_1 - T_0).$$

The function $\sigma$ was taken as a positive constant $\sigma = 0{\cdot}3$ for all calculations. This procedure yielded rapid convergence even for "very bad" initial guesses. Without this modification, the iteration method exhibits occasionally some instability. When it occurs, depends on the deviation of the initial guesses from the final results.

The numerical calculations were performed on a Remington Rand 1103 digital electronic computer.

## 4. PHYSICAL PROPERTIES

Detailed derivations of the various relationships determining transport properties of multicomponent gas mixtures as functions of mass concentration and temperature are described in [10, 11]. In this paper, we shall merely state the main assumptions regarding the properties appearing in the previous equations and summarize the working formulas. The subscript 1 in these relations refers to the injected hydrogen and subscript 2 to air.

(1) *Mixture Schmidt number*, $Sc = \nu/D_{12}$, is assumed dependent on $T$ only and is calculated according to [10].

(2) *Pure component thermal capacities*, $c_{p1}$ and $c_{p2}$, were selected from [12, 13].

(3) *Mixture viscosity ratio*, $\varphi_\mu$, is calculated by Wilke's method [11] with the following formula:

$$\varphi_\mu = \frac{\mu_1/\mu_\infty}{1 + 0{\cdot}1297\left(\dfrac{1-W}{W}\right)} + \frac{\mu_2/\mu_\infty}{1 + 3{\cdot}975\left(\dfrac{W}{1-W}\right)}.$$

The pure component viscosities, $\mu_1$ and $\mu_2$, are taken from NBS-NACA tabulation, [13]. Some extrapolation to higher temperatures was necessary in a few cases.

(4) *Mixture conductivity ratio*, $\varphi_k$, is calculated by Enskog's method, [10, 11] as follows:

$$\varphi_k = \frac{k_1/a_1 + k_2/a_2 + 0{\cdot}289\sqrt{(k_1k_2)}/a_1a_2}{\left(1 - \dfrac{0{\cdot}0209}{a_1a_2}\right)k_\infty}$$

where

$$a_1 = 1 + 0{\cdot}1886\left(\frac{1-W}{W}\right)$$

$$a_2 = 1 + 14{\cdot}228\left(\frac{W}{1-W}\right).$$

The pure component conductivities, $k_1$ and $k_2$, are calculated by Hirschfelder's method [10].

(5) *Mixture density ratio*, $\varphi_\rho$, is calculated assuming perfect-gas behavior for each component as

$$\varphi_\rho = \frac{t_\infty}{(1 + 13{\cdot}5\,W)t}.$$

(6) *Mixture thermal capacity ratio*, $\varphi_c$, is calculated for perfect gases as

$$\varphi_c = \frac{c_{p1}}{c_{p\infty}}W + \frac{c_{p2}}{c_{p\infty}}(1 - W).$$

The properties used in the present analysis do not include the effect of dissociation and chemical reactions. This had been done, because such effects are excluded from the whole analysis and since it is actually easier to generalize the results to a situation in which chemical reactions occur when transport properties excluding these effects have been used. More will be said about this point in the section on engineering correlations.

## 5. RESULTS

The following range of the parameters on which the calculations depend has been selected:

$$0 \leqslant W_w \leqslant 0{\cdot}8,\quad 0{\cdot}5 \leqslant T_w \leqslant 6,\quad 0 \leqslant M_\infty \leqslant 12,$$
$$123^\circ \leqslant t_\infty \leqslant 2000^\circ\mathrm{R}.$$

The low free-stream temperatures are indicative of high speed wind-tunnel tests. It is believed that the results of the present analysis provide also a good approximation to laminar boundary layers on flying objects in regions with small pressure gradient. On such objects, however, the air outside the boundary layer has passed a shock and is, therefore, heated to considerable temperatures. For this reason, the higher values of $t_\infty$ have been included.

Some of the skin friction results are presented in Figs. 2 and 3. The subscript zero refers to a pure air boundary layer on a solid wall ($W_w = 0$). The coefficient $c_f$ is defined in the standard way as

$$c_f = \frac{\left(\mu \dfrac{\partial u}{\partial y}\right)_w}{\frac{1}{2}\,\rho_\infty u_\infty^2}.$$

One observes a marked reduction of the skin

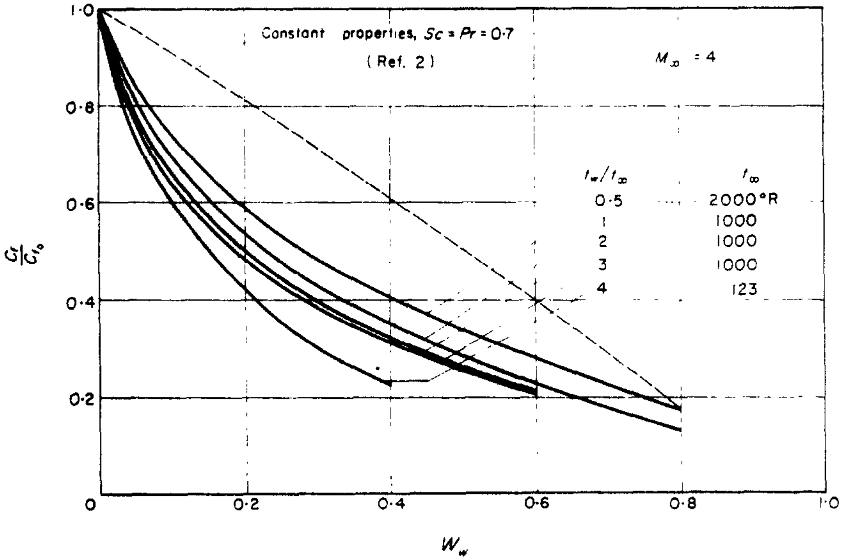

FIG. 2. Hydrogen–air, $T_w$ = const.

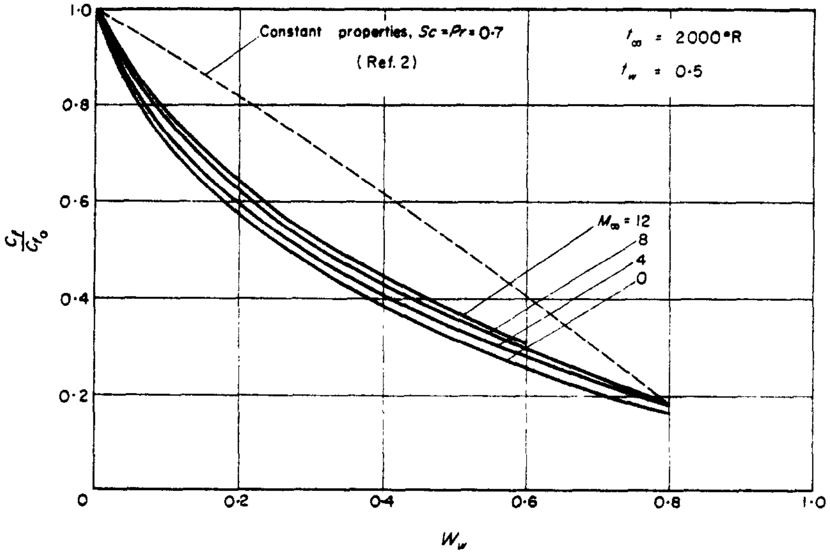

FIG. 3. Hydrogen–air, $T_w$ = const.

friction even for small hydrogen mass fraction at the wall. The reduction is larger for low values of the free-stream temperature and large values of the wall temperature, whereas the influence of Mach number is comparatively small. The dashed lines present the results of a calculation which assumes the properties to be constant and to have the same value for the main-stream fluid and for the coolant. Such calculations will approximate, for instance, situations with small Mach number, small differences between wall and free-stream temperature and with a coolant gas with properties not too different from those of the main-stream fluid. It is interesting to note that the reduction of the friction factor is, in this case, considerably smaller, especially at small values of the wall concentration.

Figures 4 and 5 present corresponding information on heat transfer to the surface. The heat flux, $q$, is defined in equation (3). It will be noted that the dependency of the heat-flux parameter, $q/q_0$, on the various boundary conditions is qualitatively the same as the one of the friction factor. Quantitatively, the parameters have a somewhat stronger influence on heat transfer than on wall friction.

Figure 6 presents the relation between the mass release parameter, $f_w$, and the coolant concentration, $W_w$, at the wall surface. It may be observed that especially the level of the free-stream temperature has a strong influence on this relation and that its character is quite different for hydrogen injection than for the constant property situation.

A recovery factor $r$ has been defined by equation (4). The ratio of this recovery factor with hydrogen injection to the recovery factor, $r_0$, on a solid wall is plotted in Fig. 7 as a function of the coolant concentration, $W_w$, at the wall surface. It is evident that the level of the free-stream temperature has a strong influence on the recovery parameter, $r/r_0$. For a free-stream temperature of 123°R, hydrogen injection increases this parameter above a value one.

## 6. ENGINEERING CORRELATIONS

Gross *et al.* [15] systematically compiled all information which was available at that time for a number of coolant gases and derived empirical relations for skin friction and heat transfer in a binary boundary layer on a "flat plate". With some minor modifications, these equations are:

$$\frac{c_f}{c_{f0}} = 1 - 2{\cdot}08 \left(\frac{M_2}{M_1}\right)^{1/3} \frac{\rho_w v_w}{\rho^{*\prime} u_\infty} \sqrt{Re_x^{*\prime}} \qquad (9)$$

$$\frac{q}{q_0} = 1 - 1{\cdot}82 \left(\frac{M_2}{M_1}\right)^{1/3} \frac{\rho_w v_w}{\rho^{*\prime} u_\infty} \sqrt{Re_x^{*\prime}}. \qquad (10)$$

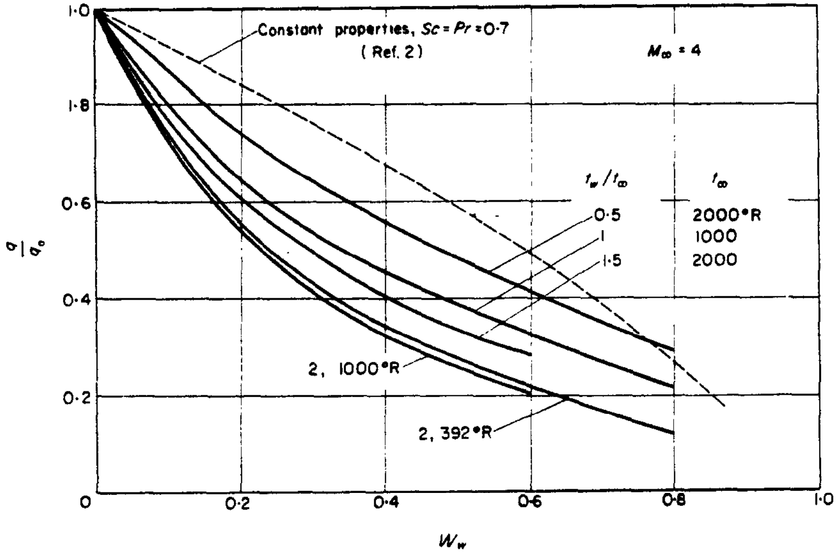

FIG. 4. Hydrogen–air, $T_w$ = const.

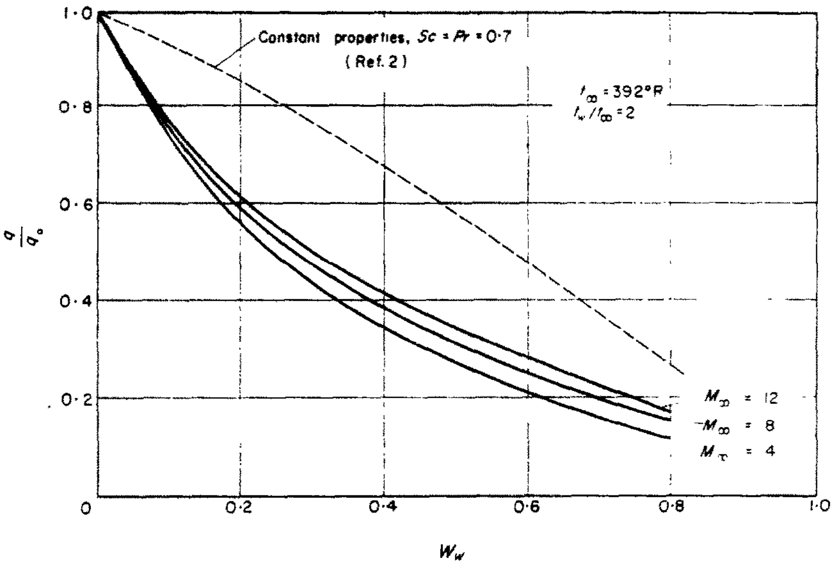

FIG. 5. Hydrogen–air, $T_w$ = const.

The star in these equations indicates that the corresponding property has to be introduced at a reference temperature

$$t^{*\prime} = t_\infty + 0{\cdot}5\,(t_w - t_\infty) + 0{\cdot}22\,(t_{r0} - t_\infty) \quad (11)$$

where the term $t_{r0}$ indicates the recovery temperature of a solid surface without mass injection but with, otherwise, identical boundary conditions. All properties in equations (9) and (10) are those for air. $M_1$ is the molecular weight of the coolant gas and $M_2$ the molecular weight of air.

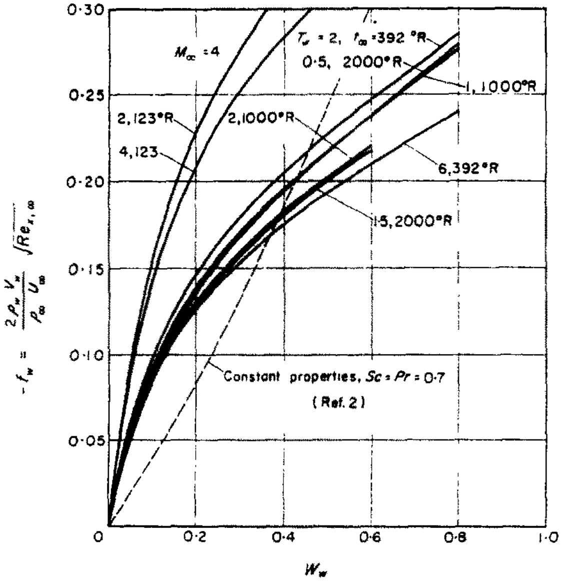

FIG. 6. Hydrogen–air, $T_w$ = const.

Only results of calculations with a free-stream temperature of 392°R were available at the time when the report [15] was prepared. The analysis in the present paper extended this parameter over a wide range of temperatures. It was, therefore, used to check the accuracy of the above correlations. Figs. 8 and 9 contain the results of our analysis as points, the symbols of which are explained in Table 1. Equations (9) and (10) are indicated as dashed lines, and dashed-dotted lines represent two other correlations which have been developed in [15] specifically for hydrogen as a coolant. The two figures indicate that the general correlations (9) and (10) somewhat underestimate the reduction in friction factor as well as in heat flux, whereas the specific correlations for hydrogen average the results of the present analysis quite well. The deviations from this relation are somewhat larger for the heat flux than for the skin friction. The relation between the mass release parameter, $f_w$, and the mass

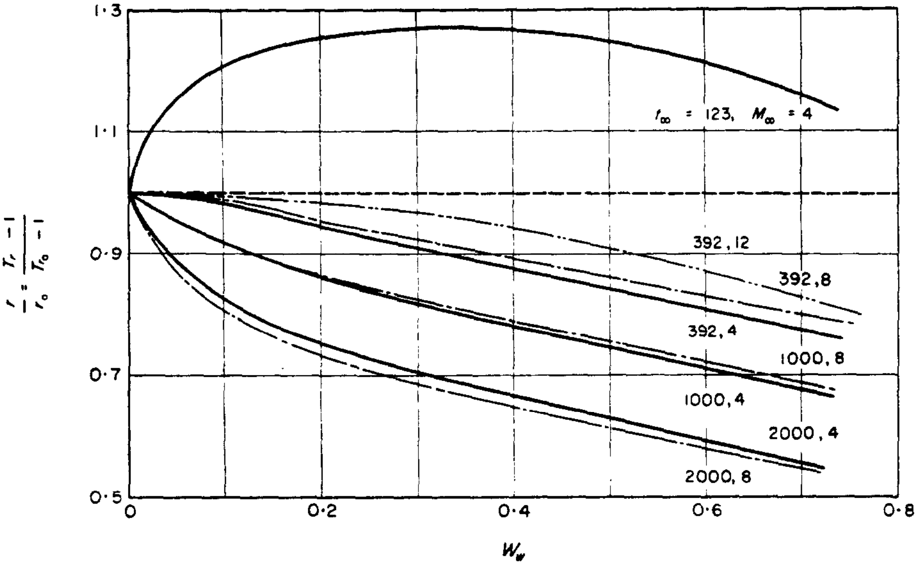

FIG. 7. Hydrogen–air, $T_w = 0$.

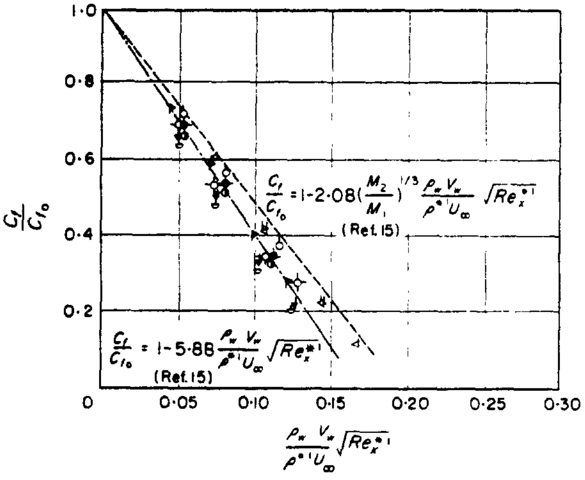

FIG. 8. Hydrogen–air, $T_w$ = const.

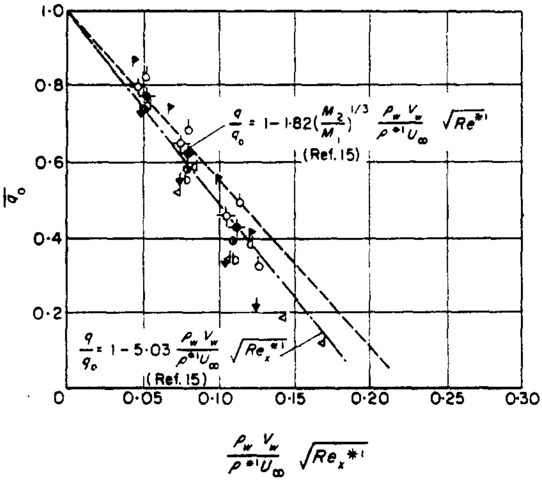

FIG. 9. Hydrogen–air, $T_w$ = const.

fraction, $W_w$, of the coolant at the wall surface is presented in Fig. 10. A correlation proposed in [15] is indicated in the figure as a solid line. Another solid line connects the analytical results for a free-stream temperature of 123°R. The correlation proposed in [15] gives a reasonable approximation to the analytical results when the values for the free-stream temperature 123°R are excluded.

In summary, it can be stated that equations (9) and (10) approximate the real conditions with an accuracy which is probably sufficient for many engineering calculations. The actual reduction in friction factor and in heat flow with hydrogen as coolant is in an average by approximately 20 per cent larger than the one predicted through these equations. A single correlation as indicated in Fig. 10 can also serve as a reasonable

*Table* 1. *Cooled wall*

| $M_\infty$ | $t_\infty$, °R | $T_w = \frac{t_w}{t_\infty}$ | Symbol |
|---|---|---|---|
| 0 | 123 | 4 | ◁ |
| 0 | 392 | 2 | ◊ |
| 0 | 392 | 6 | ♦ |
| 0 | 2000 | $\frac{1}{2}$ | ✣ |
| 4 | 123 | 4 | ▷ |
| 4 | 392 | 2 | b |
| 4 | 1000 | 1 | ⌖ |
| 4 | 1000 | 2 | ▼ |
| 4 | 1000 | 3 | ♁ |
| 4 | 2000 | $\frac{1}{2}$ | ▶ |
| 8 | 392 | 2 | d |
| 8 | 1000 | 1 | ⫰ |
| 8 | 1000 | 2 | ✦ |
| 8 | 1000 | 3 | ◐ |
| 8 | 2000 | $\frac{1}{2}$ | ◀ |
| 12 | 392 | 2 | ✚ ρ |
| 12 | 392 | 6 | ♦ |

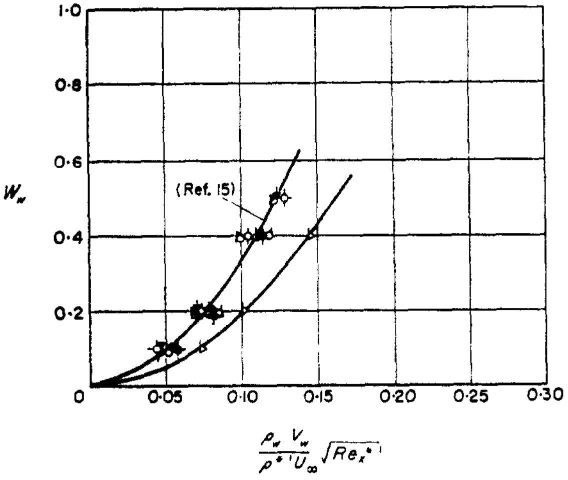

FIG. 10. Hydrogen-air, $T_w$ = const.

approximation to the relation between mass release parameter and coolant mass fraction at the wall when free-stream temperatures below approximately 300°R are excluded.

## ACKNOWLEDGEMENT

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**Résumé**—Cet article présente une étude de couche limite laminaire à deux constituants sur une surface avec gradient de pression nul, et avec perte de masse. Le système d'équations aux dérivées partielles donnant la vitesse, la température et le champ de concentration est transformé par similitude en un système d'équations intégrales. On a obtenu des solutions numériques à l'aide d'une calculatrice électronique, pour un écoulement d'air avec injection d'hydrogène pour des nombres de Mach de 0 à 12, des températures de l'écoulement libre de 123 à 2000°R et pour des rapports de température paroi/écoulement libre de ½ à 6. Les solutions dans le cas d'un gradient de température nul dans la couche limite et normal à la paroi, conduisent aux facteurs thermiques pariétaux. Les résultats sont utilisés pour étudier la validité et la précision des relations de caractère technique présentées en [15]. Les corrélations proposées décrivant assez bien la réduction du flux vers la paroi, et du frottement. Aucune corrélation simple n'a été trouvée pour l'influence de la fraction de masse d'hydrogène à la surface sur le taux de perte de masse, et pour le facteur thermique pariétal. Les valeurs obtenues pour une température d'écoulement libre de 123°R diffèrent beaucoup des autres.

**Zusammenfassung**—Die laminare zwei Komponten-Grenzschicht an einer Oberfläche ohne Druckgradienten und mit Stoffzugabe wird analysiert. Das System der partiellen Differentialgleichungen, die das Geschwindigkeits-, Temperatur- und Konzentrationsfeld beschreiben, ist durch Ähnlichkeitstransformation in ein System von Integralgleichungen übergeführt. Elektronische Digitalrechenmaschinen lieferten die numerischen Lösungen für eine Luftströmung mit Wasserstoffzugabe bei Machzahlen von 0 bis 12, Freistromtemperaturen von 68 bis 1100°K und Temperaturverhältnissen von Wand- zu Freistromtemperatur von ½ bis 6. Die Lösungen für die Bedingung eines Temperaturgradienten Null in der wandnahen Grenzschicht und senkrecht dazu führen auf Temperatur-Rückgewinnfaktoren. Mit Hilfe der Ergebnisse wird geprüft wie weit die in [15] gegebenen Gebrauchsgleichungen gültig und wie genau wie sind. Die dort vorgeschlagenen Beziehungen beschreiben die Abnahme des Wärmeflusses in die Wandoberfläche und die Oberflächenreibung verhältnismässig gut. Für die Abhängigkeit zwischen Wasserstoffanteil an der Wand und Stoffzugabe und für den Temperatur-Rückgewinnfaktor konnten keine einfachen Beziehungen gefunden werden. Die Werte für die Freistromtemperatur 68°K zeigen durchwegs die grössten Abweichungen von den anderen Ergebnissen.

**Аннотация**—Рассматривается двухкомпонентный ламинарный пограничный слой на поверхности при отсутствии градиента давления и при подаче массы. Система дифференциальных уравнений в частных производных, описывающих поля скорости, температуры и концентрации, сведена к системе интегральных уравнений. На электронно-вычислителрных машинах получены численные решения для потока воздуха с вдуванием водорода для чисел Маха от 0 до 12, температур свободного потока от 123 до 2000°R и отношений температур свободного потока и стенки от ½ до 6. Решения при усдовии равенства нулю поперечного градиента температуры на стенке дают возможность вычислить коэффициенты восстановления температуры. Результаты используются для проверки справедливости и точности инженерных формул, приведенных в работе [15]. Предложенные формулы позволяют вычислить уменьшение теплового потока в стенку и, следовательно, определить уменьшение поверхностного трения. Но ещё не удалось получить простой зависимости концентрации водорода на поверхности от скорости вдувания, а также коэффициента восстановления температуры. Для температуры свободного потока 123°R имеют место самые большие отклонения от всех остальных данных.